

TRANSPORT AND CONCENTRATION PROCESSES IN THE MULTIDIMENSIONAL ZERO-PRESSURE GAS DYNAMICS MODEL WITH THE ENERGY CONSERVATION LAW

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ABSTRACT. We introduce integral identities to define δ -shock wave type solutions for the multidimensional zero-pressure gas dynamics

$$\begin{aligned} \rho_t + \nabla \cdot (\rho U) &= 0, \\ (\rho U)_t + \nabla \cdot (\rho U \otimes U) &= 0, \\ \left(\frac{\rho |U|^2}{2} + H \right)_t + \nabla \cdot \left(\left(\frac{\rho |U|^2}{2} + H \right) U \right) &= 0, \end{aligned}$$

where where ρ is the density, $U \in \mathbb{R}^n$ is the velocity, $H(x, t)$ is the internal energy, $x \in \mathbb{R}^n$. Using these integral identities, the Rankine-Hugoniot conditions for δ -shocks are obtained. We derive the balance laws describing mass, momentum, and energy transport from the area outside the δ -shock wave front onto this front. These processes are going on in such a way that the total mass, momentum, and energy are conserved and at the same time mass and energy of the moving δ -shock wave front are increasing quantities. In addition, the total kinetic energy transfers into the total internal energy. The process of propagation of δ -shock waves is also described. These results can be used in modeling of mediums which can be treated as a *pressureless continuum* (dusty gases, two-phase flows with solid particles or droplets, granular gases).

1. STRONG SINGULAR SOLUTIONS AND PRESSURELESS MEDIUMS

1.1. **L^∞ -type solutions.** Let us recall some classical results. Consider the Cauchy problem for the system of conservation laws in one dimension space:

$$\begin{cases} U_t + (F(U))_x = 0, & \text{in } \mathbb{R} \times (0, \infty), \\ U = U^0, & \text{in } \mathbb{R} \times \{t = 0\}, \end{cases} \quad (1.1)$$

where $F : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is called the *flux-function* associated with (1.1); $U^0 : \mathbb{R} \rightarrow \mathbb{R}^m$ are given vector-functions; $U = U(x, t) = (u_1(x, t), \dots, u_m(x, t))$ is the unknown function with value in \mathbb{R}^m , and components $u_j(x, t)$, $j = 1, \dots, m$; $x \in \mathbb{R}$, $t \geq 0$.

As is well known, even in the case of smooth (and, certainly, in the case of discontinuous) initial data $U^0(x)$, in general, does not exist any smooth and global in time solution of system (1.1). As noted in the Evans' book [9, 11.1.1.], "the great difficulty in this subject is discovering a proper notion of weak solution for the initial problem (1.1)". "We must devise some way to interpret a less regular function as somehow "solving" this initial-value problem" [9, 3.4.1.a.]. But it is a well known that a partial differential equation may not make sense even if U is differentiable. "However, observe that if we *temporarily* assume U is smooth, we can as follows

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rewrite, so that the resulting expression does not directly involve the derivatives of U [9, 3.4.1.a.]. “The idea is to multiply the partial differential equation in (1.1) by a smooth function φ and then to integrate by parts, thereby transferring the derivatives onto φ ” [9, 3.4.1.a.; 11.1.1.]. Following this suggestion we shall derive the integral identity which gives the following definition of an L^∞ -generalized solution of the Cauchy problem (1.1): $U \in L^\infty(\mathbb{R} \times (0, \infty); \mathbb{R}^m)$ is called a *generalized solution* of the Cauchy problem (1.1) if the integral identity

$$\int_0^\infty \int \left(U \cdot \tilde{\varphi}_t + F(U) \cdot \tilde{\varphi}_x \right) dx dt + \int U^0(x) \cdot \tilde{\varphi}(x, 0) dx = 0 \quad (1.2)$$

holds for all compactly supported smooth test vector-functions $\tilde{\varphi} : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}^m$, where \cdot is the scalar product of vectors, and $\int f(x) dx$ denotes the improper integral $\int_{-\infty}^\infty f(x) dx$. “This identity, which we derived supposing U to be a smooth solution makes sense if U is merely bounded” [9, 11.1.1.].

Theorem 1.1. (see, e.g., [9, 11.1.1.]) *Let $\Omega \subset \mathbb{R} \times (0, \infty)$ be a region cut by a smooth curve Γ into a left- and right-hand parts Ω_\mp . Let us assume that the generalized solution U of (1.1) is smooth on either side of the curve Γ along which U has simple jump discontinuities. Then the Rankine–Hugoniot condition*

$$[F(U)]_\Gamma \nu_1 + [U]_\Gamma \nu_2 = 0, \quad (1.3)$$

holds along Γ , where $\mathbf{n} = (\nu_1, \nu_2)$ is the unit normal to the curve Γ pointing from Ω_- into Ω_+ ,

$$[F(U)] \stackrel{\text{def}}{=} F(U_-) - F(U_+),$$

$[U] \stackrel{\text{def}}{=} U_- - U_+$ are the jumps in $F(U)$ and in U across the discontinuity curve Γ , respectively. U_\mp are respective the left- and right-hand values of U on Γ .

If $\Gamma = \{(x, t) : x = \phi(t)\}$, where $\phi(\cdot) \in C^1(0, +\infty)$, then

$$\mathbf{n} = (\nu_1, \nu_2) = \frac{1}{\sqrt{1 + (\dot{\phi}(t))^2}} (1, -\dot{\phi}(t)), \quad (1.4)$$

and (1.3) reads

$$[F(U)]_\Gamma = \dot{\phi}(t) [U]_\Gamma, \quad (1.5)$$

where $\dot{(\cdot)} = \frac{d}{dt}(\cdot)$.

It is well known that if $U \in L^\infty(\mathbb{R} \times (0, \infty); \mathbb{R}^m)$ is a generalized solution of the Cauchy problem (1.1) compactly supported with respect to x , then the integral of the solution on the whole space

$$\int U(x, t) dx = \int U^0(x) dx, \quad t \geq 0 \quad (1.6)$$

is independent of time. These integrals can express the conservation laws of quantities like the *total area, mass, momentum, energy*, etc.

1.2. δ -shocks. It is well known that there are “nonclassical” situations where, in contrast to Lax’s and Glimm’s classical results, the Cauchy problem for a system of conservation laws *either does not possess a weak L^∞ -solution or possesses it for some particular initial data*. In order to solve the Cauchy problem in these “nonclassical” situations, it is necessary to seek solutions of this Cauchy problem in class of singular solutions called *δ -shocks*. Roughly speaking, a *δ -shock* is a solution such that its components *contain Dirac delta-functions*.

It is customary to assume that a δ -shock wave type solution was first described by Korchinski in his unpublished dissertation [13] in 1977. However, in fact, a solution

of this type as well as the Rankine-Hugoniot condition for the one-dimensional continuity equation were already derived from physical considerations in the book [33, §7, §12] in 1973. Next, in 1979, A. N. Kraiko [14] considered *a new type of discontinuity surface which are to be introduced in certain models of media having no inherent pressure* and obtained the Rankine-Hugoniot conditions for them. The system under consideration in [14] is the zero-pressure gas dynamics described by the system of equations:

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2)_x &= 0, \\ \left(\frac{\rho u^2}{2} + \rho \tau\right)_t + \left(\left(\frac{\rho u^2}{2} + \rho \tau\right)u\right)_x &= 0, \end{aligned} \quad (1.7)$$

where $\rho(x, t) \geq 0$ is the density, $u(x, t)$ is the velocity, $\rho(x, t)u(x, t)$ is corresponding momentum, $\tau(x, t)$ is the *internal energy per unit mass*, $x \in \mathbb{R}$. The last system can be derived from the Euler equations of nonisentropic gas dynamics

$$\rho_t + (\rho u)_x = 0, \quad (\rho u)_t + (\rho u^2 + p)_x = 0, \quad (\rho E)_t + ((\rho E + p)u)_x = 0, \quad (1.8)$$

if we set $p = 0$, where $E = \frac{\rho u^2}{2} + \tau$ is *total energy per unit mass*.

According to [14, page 502], to construct a solution for system (1.7) for arbitrary initial data, we need discontinuities which would be different from classical ones and *carry mass, impulse and energy*. As it turned out these nonclassical discontinuities are δ -shocks.

The theory of δ -shocks has been intensively developed in the last fifteen years (for example, see [2], [4]–[7], [17]–[20], [29]–[31] and the references therein). Moreover, recently, in [24], a concept of $\delta^{(n)}$ -shock wave type solutions was introduced, $n = 1, 2, \dots$. It is a *new type of singular solution* of a system of conservation laws such that its components contain delta functions and their derivatives up to n -th order. In [24], [27], the theory of δ' -shocks was established. The results [24] and [27] show that systems of conservation laws can develop solutions not only of the type of Dirac measures (as in the case of δ -shocks) but also the type of derivatives of such measures.

The above-mentioned singular solutions do not satisfy the standard integral identities of the type (1.2). To define them we use special integral identities and derive special Rankine–Hugoniot conditions. These solutions are connected with *transport and concentration processes* [2], [5], [24], [29], [28].

In the numerous papers cited above δ -shocks were studied for the *system of zero-pressure gas dynamics*:

$$\rho_t + \nabla \cdot (\rho U) = 0, \quad (\rho U)_t + \nabla \cdot (\rho U \otimes U) = 0, \quad (1.9)$$

where $\rho = \rho(x, t) \geq 0$ is the density, $U = (u_1(x, t), \dots, u_n(x, t)) \in \mathbb{R}^n$ is the velocity, $\nabla = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$, \cdot is the scalar product of vectors, \otimes is the usual tensor product of vectors.

The system of zero-pressure gas dynamics (1.9) has a physical context and is used in applications. This system can be considered as a model of the “sticky particle dynamics” and was used, e.g., to describe the formation of large-scale structures of the universe [25], [32], for modeling the formation and evolution of traffic jams [3], for modeling non-classical shallow water flows [8]. Nonlinear equations (in particular, zero-pressure gas dynamics) admitting δ -shock wave type solutions are appropriate for modeling and studying singular problems like movement of multiphase media (dusty gases, two-phase flows with solid particles or droplets). The presence of particles or droplets may drastically modify flow parameters. Moreover, a large number of phenomena that are absent in pure gas flow is inherent in two-phase flows.

Among them there are local accumulation and focusing of particles, inter-particle and particle-wall collisions resulting in particle mixing and dispersion, surface erosion due to particle impacts, and particle-turbulence interactions which govern the dispersion and concentration heterogeneities of inertial particles. The dispersed phase is usually treated mathematically as a *pressureless continuum*. Models of such media were discussed in the papers [14]–[16], [21]–[23]. Equations admitting δ -shocks can also be used for modeling granular gases. Granular gases are dilute assemblies of hard spheres which lose energy at collisions. In such gases local density excesses and local pressure falls [10], [11]. In [10], [11], the following hydrodynamics system of granular gas

$$\rho_t + (\rho u)_x = 0, \quad \rho(u_t + uu_x) = -(\rho T)_x, \quad T_t + uT_x = -(\gamma - 1)Tu_x - \Lambda \rho T^{3/2},$$

was studied, where ρ is the gas density, u is the velocity, T is the temperature, γ is the adiabatic index, $p = \rho T$ is the pressure. It was shown that for *non-zero pressure* this system admits a solution which contains a δ -function in the density ρ .

1.3. Main results. As it follows from [14]–[16], for modeling media which can be considered as *having no pressure* we must take into account energy transport. In the above-cited papers zero-pressure gas dynamics was studied only in the form (1.9). Therefore, we need to study δ -shocks in zero-pressure gas dynamics

$$\begin{aligned} \rho_t + \nabla \cdot (\rho U) &= 0, \\ (\rho U)_t + \nabla \cdot (\rho U \otimes U) &= 0, \\ \left(\frac{\rho |U|^2}{2} + H \right)_t + \nabla \cdot \left(\left(\frac{\rho |U|^2}{2} + H \right) U \right) &= 0, \end{aligned} \tag{1.10}$$

where $H(x, t)$ is the internal energy, $|U|^2 = \sum_{k=1}^n u_k^2$. This system is obtained by adding an energy conservation law to zero-pressure gas dynamics (1.9). As distinct from (1.7) it is more convenient for us to consider as a variable H instead of $H = \rho\tau$, where τ is the internal energy per unit mass. The reason is that since for singular solution $\tau(x, t)$ and $\rho(x, t)$ must contain δ -functions, it is impossible to define the product $\tau(x, t)\rho(x, t)$.

Under the second thermodynamics law it is natural to supplement the system (1.7) with a state equation $\tau = \tau(T)$, where T is the temperature. For (1.10) the natural state equation is $H = H(\rho, T)$, moreover, $H(0, T) = 0$.

In Sec. 2, we introduce Definition 2.1 of δ -shock wave type solutions for system (1.10). Next, using this definition, by Theorem 2.1 we derive the corresponding Rankine-Hugoniot conditions for δ -shocks (2.4). These Rankine-Hugoniot conditions are the direct analog of those that were introduced by A. N. Kraiko [14].

In Sec. 3, we show that δ -shocks are related with the *transport processes* of mass, momentum and energy. According to Theorems 3.1, 3.2, the mass, momentum and energy *transport processes* between the area outside of the moving δ -shock wave front and this front *are going on* such that the total mass, momentum and energy are independent of time. Moreover, the *mass and energy concentration processes* takes place on the δ -shock wave front.

2. δ -SHOCK TYPE SOLUTIONS AND THE RANKINE–HUGONIOT CONDITIONS

2.1. δ -shock type solutions. Throughout the paper we shall systematically use some results recalled in Appendix A. Let $\Gamma = \{(x, t) : S(x, t) = 0\}$ be a hypersurface of codimension 1 in the upper half-space $\{(x, t) : x \in \mathbb{R}^n, t \in [0, \infty)\} \subset \mathbb{R}^{n+1}$, $S \in C^\infty(\mathbb{R}^n \times [0, \infty))$, with $\nabla S(x, t)|_{S=0} \neq 0$ for any fixed t , where $\nabla = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$. Let $\Gamma_t = \{x \in \mathbb{R}^n : S(x, t) = 0\}$ be a moving surface in \mathbb{R}^n . Denote by ν the unit space normal to the surface Γ_t pointing (in the positive direction) from $\Omega_t^- = \{x \in$

$\mathbb{R}^n : S(x, t) < 0\}$ to $\Omega_t^+ = \{x \in \mathbb{R}^n : S(x, t) > 0\}$ such that $\nu_j = \frac{S_{x_j}}{|\nabla S|}$, $j = 1, \dots, n$. The direction of the vector ν coincides with the direction in which the function S increases, i.e., inward the domain Ω_t^+ . The time component of the normal vector $-G = \frac{S_t}{|\nabla S|}$ is the *velocity of the wave front* Γ_t along the space normal ν .

For system (1.10) we consider the *δ -shock type initial data*

$$\begin{aligned} (U^0(x), \rho^0(x), H^0(x), x \in \mathbb{R}^n; U_\delta^0(x), x \in \Gamma_0), \\ \text{where } \begin{aligned} \rho^0(x) &= \hat{\rho}^0(x) + e^0(x)\delta(\Gamma_0), \\ H^0(x) &= \hat{H}^0(x) + h^0(x)\delta(\Gamma_0), \end{aligned} \end{aligned} \quad (2.1)$$

such that $U^0 \in L^\infty(\mathbb{R}^n; \mathbb{R}^n)$, $\hat{\rho}^0, \hat{H}^0 \in L^\infty(\mathbb{R}^n; \mathbb{R})$, $e^0, h^0 \in C(\Gamma_0)$, $\Gamma_0 = \{x : S^0(x) = 0\}$ is the initial position of the δ -shock front, $\nabla S^0(x)|_{S^0=0} \neq 0$, $U_\delta^0(x)$, $x \in \Gamma_0$, is the *initial velocity* of the δ -shock, $\delta(\Gamma_0) (\equiv \delta(S^0))$ is the Dirac delta function concentrated on the surface Γ_0 defined by (A.8):

$$\langle \delta(S_0), \varphi(x) \rangle = \int_{\Gamma_0} \varphi(x) d\Gamma_0, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n),$$

$d\Gamma_0$ is the surface measure on the surface Γ_0 .

Similarly to [29, Definition 9.1.] we introduce the following definition

Definition 2.1. A triple of distributions (U, ρ, H) and a hypersurface Γ , where $\rho(x, t)$ and $H(x, t)$ have the form of the sum

$$\rho(x, t) = \hat{\rho}(x, t) + e(x, t)\delta(\Gamma), \quad H(x, t) = \hat{H}(x, t) + h(x, t)\delta(\Gamma),$$

and $U \in L^\infty(\mathbb{R}^n \times (0, \infty); \mathbb{R}^n)$, $\hat{\rho}, \hat{H} \in L^\infty(\mathbb{R}^n \times (0, \infty); \mathbb{R})$, $e, h \in C(\Gamma)$, is called a *δ -shock wave type solution* of the Cauchy problem (1.10), (2.1) if the integral identities

$$\begin{aligned} & \int_0^\infty \int \hat{\rho}(\varphi_t + U \cdot \nabla \varphi) dx dt + \int_\Gamma e \frac{\delta \varphi}{\delta t} \frac{d\Gamma}{\sqrt{1+G^2}} \\ & \quad + \int \hat{\rho}^0(x) \varphi(x, 0) dx + \int_{\Gamma_0} e^0(x) \varphi(x, 0) d\Gamma_0 = 0, \\ & \int_0^\infty \int \hat{\rho} U (\varphi_t + U \cdot \nabla \varphi) dx dt + \int_\Gamma e U_\delta \frac{\delta \varphi}{\delta t} \frac{d\Gamma}{\sqrt{1+G^2}} \\ & \quad + \int U^0(x) \hat{\rho}^0(x) \varphi(x, 0) dx + \int_{\Gamma_0} e^0(x) U_\delta^0(x) \varphi(x, 0) d\Gamma_0 = 0, \\ & \int_0^\infty \int \left(\left(\frac{\hat{\rho}|U|^2}{2} + \hat{H} \right) \varphi_t + \left(\frac{\hat{\rho}|U|^2}{2} + \hat{H} \right) U \cdot \nabla \varphi \right) dx dt \\ & \quad + \int_\Gamma \left(\frac{e|U_\delta|^2}{2} + h \right) \frac{\delta \varphi}{\delta t} \frac{d\Gamma}{\sqrt{1+G^2}} \\ & \quad + \int \left(\frac{\hat{\rho}^0(x)|U^0(x)|^2}{2} + \hat{H}^0(x) \right) \varphi(x, 0) dx \\ & \quad + \int_{\Gamma_0} \left(\frac{e^0(x)|U_\delta^0(x)|^2}{2} + h^0(x) \right) \varphi(x, 0) d\Gamma_0 = 0, \end{aligned} \quad (2.2)$$

hold for all $\varphi \in \mathcal{D}(\mathbb{R}^n \times [0, \infty))$. Here $\int f(x) dx$ denotes the improper integral $\int_{\mathbb{R}^n} f(x) dx$; $d\Gamma$ and $d\Gamma_0$ are the surface measures on the surfaces Γ and Γ_0 , respectively;

$$U_\delta = \nu G = -\frac{S_t \nabla S}{|\nabla S|^2} \quad (2.3)$$

is the δ -shock velocity, ν is the unit space normal to the surface Γ_t introduced above; $-G = \frac{S_t}{|\nabla S|}$, $\frac{\delta\varphi}{\delta t}$ is the δ -derivative with respect to the time variable (A.5); $\delta(\Gamma)$ is the Dirac delta function concentrated on the surface Γ defined by (A.8):

$$\langle \delta(S), \varphi(x, t) \rangle = \int_{-\infty}^{\infty} \int_{\Gamma_t} \varphi(x, t) d\Gamma_t dt = \int_{\Gamma} \varphi(x, t) \frac{d\Gamma}{\sqrt{1+G^2}}, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R}).$$

In view of (2.3), the δ -derivative in (2.2) can be rewritten as the Lagrangian derivative:

$$\frac{\delta\varphi}{\delta t} = \frac{\partial\varphi}{\partial t} + G \frac{\partial\varphi}{\partial\nu} = \frac{\partial\varphi}{\partial t} + U_\delta \cdot \nabla\varphi = \frac{D\varphi}{Dt}.$$

2.2. Rankine–Hugoniot conditions. Using Definition 2.1, we derive the δ -shock Rankine–Hugoniot conditions for system (1.10).

Theorem 2.1. *Let us assume that $\Omega \subset \mathbb{R}^n \times (0, \infty)$ is a region cut by a smooth hypersurface $\Gamma = \{(x, t) : S(x, t) = 0\}$ into left- and right-hand parts $\Omega^\mp = \{(x, t) : \mp S(x, t) > 0\}$. Let (U, ρ, H) , Γ be a δ -shock wave type solution of system (1.10) (in the sense of Definition 2.1), and suppose that U, ρ, H are smooth in Ω^\pm and have one-sided limits $U^\pm, \hat{\rho}^\pm, H^\pm$ on Γ . Then the Rankine–Hugoniot conditions for the δ -shock*

$$\begin{aligned} \frac{\delta e}{\delta t} + \nabla_{\Gamma_t} \cdot (eU_\delta) &= ([\rho U] - [\rho]U_\delta) \cdot \nu, \\ \frac{\delta(eU_\delta)}{\delta t} + \nabla_{\Gamma_t} \cdot (eU_\delta \otimes U_\delta) &= ([\rho U \otimes U] - [\rho U]U_\delta) \cdot \nu, \\ \frac{\delta}{\delta t} \left(\frac{e|U_\delta|^2}{2} + h \right) + \nabla_{\Gamma_t} \cdot \left(\left(\frac{e|U_\delta|^2}{2} + h \right) U_\delta \right) &= \left(\left[\left(\frac{\rho|U|^2}{2} + H \right) U \right] \right. \\ &\quad \left. - \left[\frac{\rho|U|^2}{2} + H \right] U_\delta \right) \cdot \nu, \end{aligned} \tag{2.4}$$

hold on the discontinuity hypersurface Γ , where $[f(U, \rho, H)] = f(U^-, \rho^-, H^-) - f(U^+, \rho^+, H^+)$ is the jump of the function $f(U, \rho, H)$ across the discontinuity hypersurface Γ , $\frac{\delta}{\delta t}$ is the δ -derivative (A.5) with respect to t , and ∇_{Γ_t} is defined by (A.5), (A.6).

Proof. The first two conditions in (2.4) were proved in [29, Theorem 9.1.].

Let us prove the third condition in (2.4). For any test function $\varphi \in \mathcal{D}(\Omega)$ we have $\varphi(x, t) = 0$ for $(x, t) \notin G$, $\overline{G} \subset \Omega$. Selecting the test function $\varphi(x, t)$ with compact support in Ω^\pm , we deduce from the third identity in (2.2) that the third relation in (1.10) hold in Ω^\pm , i.e.,

$$\left(\frac{\rho|U|^2}{2} + H \right)_t + \nabla \cdot \left(\left(\frac{\rho|U|^2}{2} + H \right) U \right) = 0 \quad \text{for } (x, t) \in \Omega^\pm. \tag{2.5}$$

Now, if the test function $\varphi(x, t)$ has the support in Ω , then

$$\begin{aligned} &\int_0^\infty \int \left(\left(\frac{\hat{\rho}|U|^2}{2} + \hat{H} \right) \varphi_t + \left(\frac{\hat{\rho}|U|^2}{2} + \hat{H} \right) U \cdot \nabla \varphi \right) dx dt \\ &= \int_{\Omega^- \cap G} \left(\left(\frac{\hat{\rho}|U|^2}{2} + \hat{H} \right) \varphi_t + \left(\frac{\hat{\rho}|U|^2}{2} + \hat{H} \right) U \cdot \nabla \varphi \right) dx dt \\ &\quad + \int_{\Omega^+ \cap G} \left(\left(\frac{\hat{\rho}|U|^2}{2} + \hat{H} \right) \varphi_t + \left(\frac{\hat{\rho}|U|^2}{2} + \hat{H} \right) U \cdot \nabla \varphi \right) dx dt. \end{aligned}$$

Using the integrating-by-parts formula, we obtain

$$\begin{aligned}
& \int_{\Omega^\pm \cap G} \left(\left(\frac{\hat{\rho}|U|^2}{2} + \hat{H} \right) \varphi_t + \left(\frac{\hat{\rho}|U|^2}{2} + \hat{H} \right) U \cdot \nabla \varphi \right) dx dt \\
&= - \int_{\Omega^\pm \cap G} \left(\left(\frac{\rho|U|^2}{2} + H \right)_t + \nabla \cdot \left(\left(\frac{\rho|U|^2}{2} + H \right) U \right) \right) \varphi(x, t) dx dt \\
&\mp \int_{\Gamma \cap G} \left(\left(\frac{\rho^\pm |U^\pm|^2}{2} + H^\pm \right) \frac{S_t}{|\nabla_{(x,t)} S|} + \left(\frac{\rho^\pm |U^\pm|^2}{2} + H^\pm \right) \frac{U^\pm \cdot \nabla S}{|\nabla_{(x,t)} S|} \right) \varphi(x, t) d\Gamma \\
&\quad - \int_{\Omega^\pm \cap G \cap \mathbb{R}^n} \left(\frac{\hat{\rho}^0(x) |U^0(x)|^2}{2} + \hat{H}^0(x) \right) \varphi(x, 0) dx,
\end{aligned}$$

where $d\Gamma$ is the surface measure on Γ . Next, adding the latter relations and taking into account (2.5), we have

$$\begin{aligned}
& \int_0^\infty \int \left(\left(\frac{\hat{\rho}|U|^2}{2} + \hat{H} \right) \varphi_t + \left(\frac{\hat{\rho}|U|^2}{2} + \hat{H} \right) U \cdot \nabla \varphi \right) dx dt \\
&\quad + \int \left(\frac{\hat{\rho}^0(x) |U^0(x)|^2}{2} + \hat{H}^0(x) \right) \varphi(x, 0) dx \\
&= \int_\Gamma \left(- \left[\frac{\rho|U|^2}{2} + H \right] G + \left[\left(\frac{\rho|U|^2}{2} + H \right) U \right] \cdot \nu \right) \varphi(x, t) \frac{d\Gamma}{\sqrt{1+G^2}}. \tag{2.6}
\end{aligned}$$

Next, applying the integrating-by-parts formula (A.10) to the second summand in third identity (2.2), one can see that

$$\begin{aligned}
& \int_\Gamma \left(\frac{e|U_\delta|^2}{2} + h \right) \frac{\delta \varphi}{\delta t} \frac{d\Gamma}{\sqrt{1+G^2}} + \int_{\Gamma_0} \left(\frac{e^0(x) |U_\delta^0(x)|^2}{2} + h^0(x) \right) \varphi(x, 0) d\Gamma_0 \\
&= - \int_\Gamma \frac{\delta^*}{\delta t} \left(\frac{e|U_\delta|^2}{2} + h \right) \varphi \frac{d\Gamma}{\sqrt{1+G^2}},
\end{aligned}$$

where the adjoint operator $\frac{\delta^*}{\delta t}$ is defined in (A.11). Thus

$$\begin{aligned}
& \int_\Gamma \left(\frac{e|U_\delta|^2}{2} + h \right) \frac{\delta \varphi}{\delta t} \frac{d\Gamma}{\sqrt{1+G^2}} + \int_{\Gamma_0} \left(\frac{e^0(x) |U_\delta^0(x)|^2}{2} + h^0(x) \right) \varphi(x, 0) d\Gamma_0 \\
&= - \int_\Gamma \left(\frac{\delta}{\delta t} \left(\frac{e|U_\delta|^2}{2} + h \right) + \nabla_{\Gamma_t} \cdot \left(\left(\frac{e|U_\delta|^2}{2} + h \right) G \nu \right) \right) \varphi \frac{d\Gamma}{\sqrt{1+G^2}}. \tag{2.7}
\end{aligned}$$

Adding (2.6) and (2.7) and taking into account (2.2), (2.3), we derive

$$\begin{aligned}
& \int_\Gamma \left(- \left[\frac{\rho|U|^2}{2} + H \right] U_\delta \cdot \nu + \left[\left(\frac{\rho|U|^2}{2} + H \right) U \right] \cdot \nu \right. \\
&\quad \left. - \frac{\delta}{\delta t} \left(\frac{e|U_\delta|^2}{2} + h \right) - \nabla_{\Gamma_t} \cdot \left(\left(\frac{e|U_\delta|^2}{2} + h \right) U_\delta \right) \right) \varphi(x, t) \frac{d\Gamma}{\sqrt{1+G^2}} = 0,
\end{aligned}$$

for all $\varphi \in \mathcal{D}(\Omega)$. Thus, the third relation in (2.4) holds. \square

The right-hand sides of the equations in (2.4) are called the *Rankine–Hugoniot deficits* in ρ , ρU , and $\frac{\rho|U|^2}{2} + H$, respectively.

Let $a(x, t)$ be a smooth function defined only on the surface $\Gamma = \{(x, t) : S(x, t) = 0\}$ which is the restriction of some smooth function defined in a neighborhood of Γ in $\mathbb{R}^n \times \mathbb{R}$. It is easy to prove that

$$\nabla_{\Gamma_t} \cdot (a U_\delta) = -2\mathcal{K}Ga, \tag{2.8}$$

where \mathcal{K} is the mean curvature of the surface Γ_t (see (A.7)). Indeed, according to (A.5), (A.6), (A.7), (2.3), we have $\nabla_{\Gamma_t} \cdot (aU_\delta) = \sum_{k=1}^n \frac{\delta(Ga\nu_k)}{\delta x_k} = \sum_{k=1}^n \frac{\delta(Ga)}{\delta x_k} \nu_k + Ga \sum_{k=1}^n \frac{\delta \nu_k}{\delta x_k} = -2\mathcal{K}Ga$. Here the obvious relation $\sum_{k=1}^n \frac{\delta(Ga)}{\delta x_k} \nu_k = 0$ was taken into account.

Due to (2.8), the Rankine–Hugoniot conditions (2.4) can also be rewritten as

$$\begin{aligned} \frac{\delta e}{\delta t} - 2\mathcal{K}Ge &= ([\rho U] - [\rho]U_\delta) \cdot \nu, \\ \frac{\delta(eU_\delta)}{\delta t} - 2\mathcal{K}GeU_\delta &= ([\rho U \otimes U] - [\rho U]U_\delta) \cdot \nu, \\ \frac{\delta}{\delta t} \left(\frac{e|U_\delta|^2}{2} + h \right) - 2\mathcal{K}G \left(\frac{e|U_\delta|^2}{2} + h \right) &= \left(\left[\left(\frac{\rho|U|^2}{2} + H \right) U \right] \right. \\ &\quad \left. - \left[\frac{\rho|U|^2}{2} + H \right] U_\delta \right) \cdot \nu. \end{aligned} \quad (2.9)$$

Remark 2.1. The Rankine–Hugoniot conditions (2.4) constitute a system of *second-order* PDEs. According to this fact, for system (1.10) we use the initial data (2.1) which contain the *initial velocity* $U_\delta^0(x)$ of a δ -shock. This is similar to the fact that in the *measure-valued solution* approach [4], [17], [18], [31] the velocity U is determined on the discontinuity surface.

In the direction ν the characteristic equation of system (1.10) has repeated eigenvalues $\lambda = U \cdot \nu$. So, we assume that for the initial data (2.1) the *geometric entropy condition* holds:

$$U^{0+}(x) \cdot \nu^0|_{\Gamma_0} < U_\delta^0(x) \cdot \nu^0|_{\Gamma_0} < U^{0-}(x) \cdot \nu^0|_{\Gamma_0}, \quad (2.10)$$

where $\nu^0 = \frac{\nabla S^0(x)}{|\nabla S^0(x)|}$ is the unit space normal of Γ_0 , oriented from $\Omega_0^- = \{x \in \mathbb{R}^n : S^0(x) < 0\}$ to $\Omega_0^+ = \{x \in \mathbb{R}^n : S^0(x) > 0\}$. Similarly, we assume that for a solution of the Cauchy problem (1.10), (2.1) the *geometric entropy condition* holds:

$$U^+(x, t) \cdot \nu|_{\Gamma_t} < U_\delta(x, t) \cdot \nu|_{\Gamma_t} < U^-(x, t) \cdot \nu|_{\Gamma_t}, \quad (2.11)$$

where U_δ is the velocity (2.3) of the δ -shock front Γ_t , U^\pm is the velocity behind the δ -shock wave front and ahead of it, respectively. Condition (2.11) implies that all characteristics on both sides of the discontinuity Γ_t must overlap. For $t = 0$ the condition (2.11) coincides with (2.10).

3. δ -SHOCK MASS, MOMENTUM AND ENERGY TRANSPORT RELATIONS

The classical conservation laws (1.6) do not make sense for a δ -shock wave type solution. “Generalized” analogs of conservation laws (1.6) were derived in [2], [24], [28] for the one-dimensional case, and in [29] for the multidimensional case. Now we derive these transport conservation laws for the case of system (1.10).

Let us assume that a moving surface $\Gamma_t = \{x : S(x, t) = 0\}$ permanently separates \mathbb{R}_x^n into two parts $\Omega_t^\pm = \{x \in \mathbb{R}^n : \pm S(x, t) > 0\}$, and $\Omega_0^\pm = \{x \in \mathbb{R}^n : \pm S^0(x) > 0\}$. Let (U, ρ, H) be compactly supported with respect to x . Denote by

$$M(t) = \int_{\Omega_t^- \cup \Omega_t^+} \rho(x, t) dx, \quad m(t) = \int_{\Gamma_t} e(x, t) d\Gamma_t, \quad (3.1)$$

and

$$P(t) = \int_{\Omega_t^- \cup \Omega_t^+} \rho(x, t) U(x, t) dx, \quad p(t) = \int_{\Gamma_t} e(x, t) U_\delta(x, t) d\Gamma_t, \quad (3.2)$$

masses and momenta of the volume $\Omega_t^- \cup \Omega_t^+$ and the moving δ -shock wave front Γ_t , respectively, $d\Gamma_t$ being the surface measure on Γ_t . Let

$$W_{kin}(t) = \int_{\Omega_t^- \cup \Omega_t^+} \frac{\rho(x,t)|U(x,t)|^2}{2} dx, \quad w_{kin}(t) = \int_{\Gamma_t} \frac{e(x,t)|U_\delta(x,t)|^2}{2} d\Gamma_t, \quad (3.3)$$

and

$$W_{int}(t) = \int_{\Omega_t^- \cup \Omega_t^+} H(x,t) dx, \quad w_{int}(t) = \int_{\Gamma_t} h(x,t) d\Gamma_t, \quad (3.4)$$

be the kinetic and internal energies of the volume $\Omega_t^- \cup \Omega_t^+$ and the moving wave front Γ_t , respectively. Here $W_{kin}(t) + w_{kin}(t)$ and $W_{int}(t) + w_{int}(t)$ are the total kinetic and internal energies, respectively; $W_{kin}(t) + w_{kin}(t) + W_{int}(t) + w_{int}(t)$ is the total energy.

Theorem 3.1. *Let (U, ρ, H) together with a discontinuity hypersurface $\Gamma = \{(x, t) : S(x, t) = 0\}$ be a δ -shock wave type solution (in the sense of Definition 2.1) of the Cauchy problem (1.10), (2.1), where*

$$\rho(x, t) = \hat{\rho}(x, t) + e(x, t)\delta(\Gamma), \quad H(x, t) = \hat{H}(x, t) + h(x, t)\delta(\Gamma).$$

Let this solution satisfy the entropy condition (2.11). Suppose that (U, ρ, H) is compactly supported with respect to x , smooth in $\Omega^\pm = \{(x, t) : \pm S(x, t) > 0\}$ and has one-sided limits $U^\pm, \hat{\rho}^\pm, \hat{H}^\pm$ on Γ . Then the following mass and momentum balance relations hold:

$$\begin{aligned} \dot{M}(t) &= -\dot{m}(t), & \dot{m}(t) &\geq 0, & \dot{P}(t) &= -\dot{p}(t), \\ M(t) + m(t) &= M(0) + m(0), & P(t) + p(t) &= P(0) + p(0). \end{aligned} \quad (3.5)$$

In fact, the proof of Theorem 3.1 coincides with the proof of [29, Theorem 9.2.]. The proof of [29, Theorem 9.2.], and, consequently, the proof of Theorem 3.1 are based on the volume and surface transport Theorems A.1, A.2 and use the first two relations in (2.8).

Theorem 3.2. *Let (U, ρ, H) together with a discontinuity hypersurface $\Gamma = \{(x, t) : S(x, t) = 0\}$ satisfy the same conditions as in Theorem 3.1. Then the following energy balance relations hold:*

$$\dot{w}_{kin}(t) + \dot{w}_{int}(t) \geq 0, \quad \dot{W}_{kin}(t) \leq 0, \quad \dot{W}_{int}(t) + \dot{w}_{int}(t) \geq 0, \quad \dot{W}_{int}(t) \leq 0. \quad (3.6)$$

Moreover,

$$\begin{aligned} \dot{W}_{kin}(t) + \dot{w}_{kin}(t) &= -(\dot{W}_{int}(t) + \dot{w}_{int}(t)), \\ W_{kin}(t) + w_{kin}(t) + W_{int}(t) + w_{int}(t) &= \\ &= W_{kin}(0) + w_{kin}(0) + W_{int}(0) + w_{int}(0). \end{aligned} \quad (3.7)$$

Proof. 1. Let us assume that the supports of $U(x, t)$ and $\rho(x, t)$ with respect to x belong to a compact $K \in \mathbb{R}_x^n$ bounded by ∂K . Let $K_t^\pm = \Omega_t^\pm \cap K$. By ν we denote, as before, the space normal to Γ_t pointing from Ω_t^- to Ω_t^+ . Differentiating $W_{kin}(t) + W_{int}(t)$ and using the volume transport Theorem A.1, we obtain

$$\begin{aligned} \dot{W}_{kin}(t) + \dot{W}_{int}(t) &= \int_{K_t^- \cup K_t^+} \frac{\partial}{\partial t} \left(\frac{\rho(x, t)|U(x, t)|^2}{2} + H(x, t) \right) dx \\ &+ \int_{\partial K_t^- \cup \partial K_t^+} \left(\frac{\rho(x, t)|U(x, t)|^2}{2} + H(x, t) \right) V(x, t) \cdot \tilde{\nu} d\Gamma_t, \end{aligned} \quad (3.8)$$

where $\tilde{\nu}$ is the outward unit space normal to the surface ∂K_t^\pm and $V(x, t)$ is the velocity of the point x in K_t^\pm .

Next, taking into account that for $x \in K_t^\pm$ system (1.10) has a smooth solution (U^\pm, ρ^\pm, H^\pm) , i.e.,

$$\left(\frac{\rho^\pm |U^\pm|^2}{2} + H^\pm\right)_t + \nabla \cdot \left(\left(\frac{\rho^\pm |U^\pm|^2}{2} + H^\pm\right)U^\pm\right) = 0,$$

and U^\pm, ρ^\pm, H^\pm are equal to zero on the hypersurface ∂K_t^\pm except Γ_t , applying Gauss's divergence theorem to relation (3.8), we transform it to the form

$$\begin{aligned} \dot{W}_{kin}(t) + \dot{W}_{int}(t) &= - \int_{K_t^-} \nabla \cdot \left(\left(\frac{\rho^- |U^-|^2}{2} + H^-\right)U^-\right) dx \\ &\quad - \int_{K_t^+} \nabla \cdot \left(\left(\frac{\rho^+ |U^+|^2}{2} + H^+\right)U^+\right) dx + \int_{\Gamma_t} \left[\frac{\rho |U|^2}{2} + H\right] U_\delta \cdot \nu d\Gamma_t \\ &= - \int_{\Gamma_t} \left(\frac{\rho^- |U^-|^2}{2} + H^-\right) U^- \cdot \nu d\Gamma_t \\ &\quad + \int_{\Gamma_t} \left(\frac{\rho^+ |U^+|^2}{2} + H^+\right) U^+ \cdot \nu d\Gamma_t + \int_{\Gamma_t} \left[\frac{\rho |U|^2}{2} + H\right] U_\delta \cdot \nu d\Gamma_t \\ &= - \int_{\Gamma_t} \left(\left[\left(\frac{\rho |U|^2}{2} + H\right)U\right] - \left[\frac{\rho |U|^2}{2} + H\right]U_\delta\right) \cdot \nu d\Gamma_t, \end{aligned} \quad (3.9)$$

where $U_\delta = V|_{\Gamma_t}$ is the velocity (2.3) of the δ -shock front Γ_t . Using the third Rankine–Hugoniot condition (2.4), relation (3.9) can be rewritten as

$$\begin{aligned} \dot{W}_{kin}(t) + \dot{W}_{int}(t) &= - \int_{\Gamma_t} \left(\frac{\delta}{\delta t} \left(\frac{e|U_\delta|^2}{2} + h\right) + \nabla_{\Gamma_t} \cdot \left(\left(\frac{e|U_\delta|^2}{2} + h\right)U_\delta\right)\right) d\Gamma_t. \end{aligned} \quad (3.10)$$

Applying the surface transport Theorem A.2 to the second relations in (3.3), (3.4) one can see that the right-hand side of (3.10) coincides with $-\dot{w}_{kin}(t) - \dot{w}_{int}(t)$. Thus relations (3.7) hold.

Since $\rho^\pm \geq 0$, $H^\pm \geq 0$ and the solution (U, ρ, H) of the Cauchy problem (1.10), (2.1) satisfies the entropy condition (2.11), we have

$$\begin{aligned} &([\rho |U|^2 U] - [\rho |U|^2]U_\delta) \cdot \nu \\ &= (\rho^- |U^-|^2 (U^- - U_\delta) \cdot \nu + \rho^+ |U^+|^2 (U_\delta - U^+) \cdot \nu)|_{\Gamma_t} \geq 0; \end{aligned} \quad (3.11)$$

$$([HU] - [H]U_\delta) \cdot \nu = (H^- (U^- - U_\delta) \cdot \nu + H^+ (U_\delta - U^+) \cdot \nu)|_{\Gamma_t} \geq 0. \quad (3.12)$$

Formulas (3.9), (3.11), (3.12) imply that $\dot{W}_{kin}(t) + \dot{W}_{int}(t) \leq 0$, i.e., due to (3.7) the first inequality in (3.6) holds.

2. In fact, the second inequality in (3.6) was proved in [26].

Let us calculate $\dot{w}(t)$. Taking into account formula (2.8), due to the surface transport Theorem A.2, we obtain

$$\begin{aligned} \dot{w}_{kin}(t) &= \frac{1}{2} \int_{\Gamma_t} \left(\frac{\delta}{\delta t} (e(x, t) |U_\delta(x, t)|^2) + \nabla_{\Gamma_t} \cdot (e(x, t) |U_\delta(x, t)|^2 U_\delta)\right) d\Gamma_t \\ &= \frac{1}{2} \int_{\Gamma_t} \left(\frac{\delta}{\delta t} (e(x, t) |U_\delta(x, t)|^2) - 2\mathcal{K}G e(x, t) |U_\delta(x, t)|^2\right) d\Gamma_t \\ &= \frac{1}{2} \int_{\Gamma_t} \left(\sum_{k=1}^n \left(u_{\delta k} \frac{\delta(e u_{\delta k})}{\delta t} + u_{\delta k} e \frac{\delta u_{\delta k}}{\delta t}\right) - 2\mathcal{K}G e(x, t) |U_\delta(x, t)|^2\right) d\Gamma_t. \end{aligned} \quad (3.13)$$

According to (2.9) and (2.8), we have

$$\begin{aligned} \frac{\delta e}{\delta t} u_{\delta k} + e \frac{\delta u_{\delta k}}{\delta t} - 2\mathcal{K}Ge u_{\delta k} &= [\rho u_k U \cdot \nu] - [\rho u_k] U_\delta \cdot \nu, \\ \frac{\delta e}{\delta t} u_{\delta k} - 2\mathcal{K}Ge u_{\delta k} &= [\rho U \cdot \nu] u_{\delta k} - [\rho] U_\delta \cdot \nu u_{\delta k}, \end{aligned} \quad (3.14)$$

where $u_{\delta k}(x, t)$ is the k -th component of the vector U_δ , $k = 1, \dots, n$. Now, subtracting one equation from the other in (3.14), we obtain

$$e \frac{\delta u_{\delta k}}{\delta t} = [\rho u_k U \cdot \nu] - [\rho u_k] U_\delta \cdot \nu - [\rho U \cdot \nu] u_{\delta k} + [\rho] U_\delta \cdot \nu u_{\delta k}. \quad (3.15)$$

Substituting equations (3.15) into (3.13), one can easily calculate

$$\begin{aligned} \dot{w}_{kin}(t) &= \frac{1}{2} \int_{\Gamma_t} \left(2 \sum_{k=1}^n ([\rho u_k U \cdot \nu] - [\rho u_k] U_\delta \cdot \nu) u_{\delta k} \right. \\ &\quad \left. - [\rho U \cdot \nu] |U_\delta|^2 + [\rho] |U_\delta|^2 U_\delta \cdot \nu \right) d\Gamma_t. \end{aligned}$$

Taking into account that $U_\delta = G\nu$, $G = -\frac{S_t}{|\nabla S|}$, i.e., $u_{\delta k} = G\nu_k$, $k = 1, \dots, n$, we rewrite the above relation as

$$\dot{w}_{kin}(t) = \frac{1}{2} \int_{\Gamma_t} \left(2[\rho(U \cdot \nu)^2]G - 3[\rho U \cdot \nu]G^2 + [\rho]G^3 \right) d\Gamma_t. \quad (3.16)$$

Using (3.9) and (3.16), we obtain

$$\begin{aligned} \dot{W}_{kin}(t) + \dot{w}_{kin}(t) &= -\frac{1}{2} \int_{\Gamma_t} ([\rho|U|^2 U \cdot \nu] - [\rho|U|^2] U_\delta \cdot \nu \\ &\quad - 2[\rho(U \cdot \nu)^2]G + 3[\rho U \cdot \nu]G^2 - [\rho]G^3) d\Gamma_t. \end{aligned} \quad (3.17)$$

The gas velocity $U|_{\Gamma_t}$ on the wave front Γ_t is the sum of the normal component $U \cdot \nu$ and the component U_{tan} tangential to the surface Γ_t . Since $|U|^2|_{\Gamma_t} = (U \cdot \nu)^2 + U_{tan}^2$, and $G = U_\delta \cdot \nu$, one can represent the integrand in (3.17) as

$$\begin{aligned} &([\rho|U|^2 U \cdot \nu] - [\rho|U|^2] U_\delta \cdot \nu - 2[\rho(U \cdot \nu)^2]G + 3[\rho U \cdot \nu]G^2 - [\rho]G^3) \\ &= \rho^-(U_{tan}^-)^2(U^- \cdot \nu - U_\delta \cdot \nu) + \rho^+(U_{tan}^+)^2(U_\delta \cdot \nu - U^+ \cdot \nu) \\ &\quad + \rho^-(U^- \cdot \nu - U_\delta \cdot \nu)^3 + \rho^+(U_\delta \cdot \nu - U^+ \cdot \nu)^3. \end{aligned} \quad (3.18)$$

Since a solution (U, ρ, H) of the Cauchy problem (1.10), (2.1) satisfies the entropy condition (2.11) and $\rho^\pm \geq 0$, we deduce that the expression (3.18) is *non-negative*. Formulas (3.17), (3.18) imply that $\dot{W}_{kin}(t) + \dot{w}_{kin}(t) \leq 0$. Due to (3.7), we conclude that the third inequality in (3.6) holds.

3. Since U, ρ, H are smooth in $\Omega^\pm = \{(x, t) : \pm S(x, t) > 0\}$, it easy to see that for $(x, t) \in K_t^\pm$ the first and second equations in (1.10) imply that

$$(u_k^\pm)_t + \sum_{j=1}^n u_j \frac{\partial u_k^\pm}{\partial x_j} = 0, \quad k = 1, 2, \dots, n.$$

Multiplying the both sides of the above equation by u_k^\pm and summarizing over $k = 1, 2, \dots, n$, we obtain

$$(|U^\pm|^2)_t + \sum_{j=1}^n u_j^\pm \frac{\partial (|U^\pm|^2)}{\partial x_j} = 0, \quad (x, t) \in K_t^\pm. \quad (3.19)$$

According to (3.19) and the first equation in (1.10)

$$\left(\frac{\rho^\pm |U^\pm|^2}{2} \right)_t + \sum_{j=1}^n \frac{\partial}{\partial x_j} \left(\frac{\rho^\pm |U^\pm|^2}{2} u_j^\pm \right) = 0, \quad (x, t) \in K_t^\pm. \quad (3.20)$$

In the end, from (3.20) and the third equation in (1.10) we obtain that

$$(H^\pm)_t + \nabla \cdot (H^\pm U^\pm) = 0, \quad (x, t) \in K_t^\pm. \quad (3.21)$$

Next, as before, differentiating $W_{int}(t)$, using (3.21) and applying the volume transport Theorem A.1, we obtain

$$\begin{aligned} \dot{W}_{int}(t) &= \int_{K_t^- \cup K_t^+} \frac{\partial H(x, t)}{\partial t} dx + \int_{\partial K_t^- \cup \partial K_t^+} H(x, t) V(x, t) \cdot \tilde{\nu} d\Gamma_t, \\ &= - \int_{K_t^- \cup K_t^+} \nabla \cdot (HU) dx + \int_{\partial K_t^- \cup \partial K_t^+} H(x, t) V(x, t) \cdot \tilde{\nu} d\Gamma_t, \end{aligned} \quad (3.22)$$

where $\tilde{\nu}$ is the outward unit space normal to the surface ∂K_t^\pm and $V(x, t)$ is the velocity of the point x in K_t^\pm .

Taking into account that U^\pm, ρ^\pm, H^\pm are equal to zero on the hypersurface ∂K_t^\pm except Γ_t and applying Gauss's divergence theorem to (3.22), we transform this relation to the form

$$\begin{aligned} \dot{W}_{int}(t) &= - \int_{K_t^-} \nabla \cdot (H^- U^-) dx - \int_{K_t^+} \nabla \cdot (H^+ U^+) dx + \int_{\Gamma_t} [H] U_\delta \cdot \nu d\Gamma_t \\ &= - \int_{\Gamma_t} H^- U^- \cdot \nu d\Gamma_t + \int_{\Gamma_t} H^+ U^+ \cdot \nu d\Gamma_t + \int_{\Gamma_t} [H] U_\delta \cdot \nu d\Gamma_t \\ &= - \int_{\Gamma_t} ([HU] - [H] U_\delta) \cdot \nu d\Gamma_t, \end{aligned} \quad (3.23)$$

where $U_\delta = V|_{\Gamma_t}$ is the velocity (2.3) of the δ -shock front Γ_t , $\nu = \tilde{\nu}|_{\Gamma_t}$ is the space normal to Γ_t pointing from K_t^- to K_t^+ . In view of the entropy condition (2.11) the inequality (3.12) holds, and consequently, (3.23) implies the fourth inequality in (3.6). \square

Corollary 3.1. *According to Theorems 3.1, 3.2, the mass, momentum and energy transport processes between the volume outside of the δ -shock wave front $\Omega_t^- \cup \Omega_t^+ = \{x \in \mathbb{R}^n : S(x, t) \neq 0\}$ and the moving δ -shock wave front Γ_t are going on such that the total mass $M(t) + m(t)$, momentum $P(t) + p(t)$ and energy $W_{kin}(t) + w_{kin}(t) + W_{int}(t) + w_{int}(t)$ are independent of time. More precisely the mass and energy concentration processes on the moving δ -shock wave front Γ_t are going on. In addition, the total kinetic energy $W_{kin}(t) + w_{kin}(t)$ transfers into the total internal energy $W_{int}(t) + w_{int}(t)$.*

The inequality $\dot{W}(t) \leq 0$ in (3.6) reflects the well-known fact that the evolution of a solution with shocks is connected with decreasing of the kinetic energy.

Remark 3.1. Let us suppose that in a finite time \tilde{t} the whole initial mass $M(0)$ and energy $W_{kin}(0) + W_{int}(0)$ may be concentrated on the δ -shock front $\Gamma_{\tilde{t}}$. Then, according to The Rankine–Hugoniot conditions, for $t > \tilde{t}$, instead of the whole system of zero-pressure gas dynamics (1.10) we obtain exactly a “surface” version of this system

$$\begin{aligned} \frac{\delta e}{\delta t} + \nabla_\Gamma \cdot (e U_\delta) &= 0, \\ \frac{\delta(e U_\delta)}{\delta t} + \nabla_\Gamma \cdot (e U_\delta \otimes U_\delta) &= 0, \\ \frac{\delta}{\delta t} \left(\frac{e |U_\delta|^2}{2} + h \right) + \nabla_{\Gamma_t} \cdot \left(\left(\frac{e |U_\delta|^2}{2} + h \right) U_\delta \right) &= 0, \end{aligned}$$

where U_δ is the velocity of the δ -shock front Γ_t , e is the surface density of the front mass, h is the surface density of the front internal energy. This system is an analog

of the initial system of zero-pressure gas dynamics (1.10) on the $(n-1)$ -dimensional surface Γ_t . This $(n-1)$ -dimensional analog also has the same type as the initial system, therefore its solution can develop singularities within a finite time, and all mass concentrates on the manifold of dimension $n-2$, and so on. Thus, it may happen that after the finite number of bifurcations the whole initial mass will be concentrated at the singular point.

4. EXAMPLE OF AN ONE DIMENSIONAL CONCENTRATION PROCESS

In the 1D case we construct an explicit example of the concentration process based on another method. Namely, let us consider the data that do not imply the δ -shock initially:

$$\begin{aligned} U^0(x) &= (U^- - [U]\Theta(x))\chi_I(x), \\ \rho^0(x) &= (\rho^- - [\rho]\Theta(x))\chi_I(x), \\ H^0(x) &= (H^- - [H]\Theta(x))\chi_I(x), \end{aligned} \quad (4.1)$$

where $U^-, \rho^-, H^-, U^+, \rho^+, H^+$ are constants, $\rho^-, \rho^+, H^-, H^+ > 0$, $[U] > 0$, $\chi_I(x)$ is the characteristic function of the segment $I = [-L, L]$, $L \gg 1$. Let us note that we can apply the standard mollification procedure to obtain functions smooth at the points $\pm L$, but here do not need to do it.

We obtain the solution to the Cauchy problem by means of the free particles method [1]: first we assume that the particles do not feel one others and form the overlapping domain. Then we switch to the sticky particles model and change this overlapping domain to a point where the mass accumulates according to the conservation of mass and momentum. Now we have to consider the additional law of conservation of energy. Thus, according to ([1]), the free-particles solution (ρ_{FP}, U_{FP}) to the two first equations to the zero pressure model has the form

$$\begin{aligned} \rho_{FP}(t, x) &= \begin{cases} \rho^-, & -L + (U^- - [U])t < x < (U^- - [U])t, \\ 2\rho^- - [\rho], & (U^- - [U])t < x < U^-t, \\ \rho^- - [\rho], & L + U^-t > x > U^-t, \\ 0, & \text{otherwise,} \end{cases} \\ U_{FP}(t, x) &= \begin{cases} U^-, & -L + (U^- - [U])t < x < (U^- - [U])t, \\ U^- + \frac{\rho^- - [\rho]}{2\rho^- - [\rho]}[U], & (U^- - [U])t < x < U^-t, \\ U^- - [U], & L + U^-t > x > U^-t. \end{cases} \end{aligned}$$

Outside of the segment $[-L + (U^- - [U])t, L + U^-t]$ the solution U_{FP} contains a rarefaction wave, however this part of solution does not contribute to the energy, since for the domain of rarefaction $\rho = H = 0$. The respective solution (ρ, U) to the sticky particles model is

$$\begin{aligned} \rho(x, t) &= \rho^- - [\rho]\Theta(x - x_j(t)) + e(t)\delta(x - x_j(t)), \\ U(x, t) &= U^- - [U]\Theta(x - x_j(t)), \end{aligned}$$

where the position of the singularity $x_j(t)$ is the following:

$$x_j(t) = \frac{[U\rho] - \sqrt{(-[U\rho]^2 + [\rho][U^2\rho])}}{[\rho]}t, \quad \text{if } [\rho] \neq 0, \quad (4.2)$$

and

$$x_j(t) = \frac{2U^- - [U]}{2}t = \frac{U^- + U^+}{2}t, \quad \text{if } [\rho] = 0. \quad (4.3)$$

The amplitude of the δ -shock reads as

$$e(t) = [U\rho]t - [\rho]x_j(t).$$

The solution induces the following balance of energy.

For the sake of simplicity we dwell on the latter case. Thus,

$$\begin{aligned}
W_{int}(t) &= (x_j(t) + L - U^-t)H^- + (L + (U^- - [U])t - x_j(t))H^+ \\
&= W_{int}(0) - \frac{H^+ + H^-}{2} [U] t, \\
W_{kin}(t) &= \frac{1}{2} \left((x_j(t) + L - U^-t)\rho^-(U^-)^2 \right. \\
&\quad \left. + (L + (U^- - [U])t - x_j(t))\rho^+(U^+)^2 \right) \\
&= W_{kin}(0) - \rho[U] \frac{(U^1)^2 + (U^2)^2}{4} t, \\
w_{kin} &= \frac{1}{2} [U] \rho \left(\frac{(U^1) + (U^2)}{2} \right)^2 t, \\
w_{int} &= W_{int}(0) + W_{kin}(0) - W_{int}(t) - W_{kin}(t) - w_{kin}(t) = \\
&\quad \frac{[U]}{2} \left(\frac{\rho[U]^2}{4} + (H^- + H^+) \right) t.
\end{aligned}$$

We see that $W_{kin}(t)$ and $W_{int}(t)$ decrease with a constant velocity, and vanish within a finite time, $w_{kin}t$ increases unless $U^- + U^+ \neq 0$, (in the latter case $w_{kin}(t) \equiv 0$), $w_{int}(t)$ increases *in any case*. Since we associate the internal energy with a temperature, it signifies that the concentration process *always* entails the heating of point of the mass accumulation and cooling-down of the environment to the "absolute zero" that relates to the zero internal energy.

APPENDIX A. SOME AUXILIARY FACTS

A.1. Moving surfaces of discontinuity. Let us present some results from [12, 5.2.] concerning moving surfaces. Let Γ_t be a smooth moving surface of codimension 1 in the space \mathbb{R}^n . Such a surface can be represented locally either in the form $\Gamma_t = \{x \in \mathbb{R}^n : S(x, t) = 0\}$, or in terms of the curvilinear Gaussian coordinates $s = (s_1, \dots, s_{n-1})$ on the surface:

$$x_j = x_j(s_1, \dots, s_{n-1}, t), \quad s \in \mathbb{R}^{n-1}.$$

We also consider the surface $\Gamma = \{(x, t) \in \mathbb{R}^{n+1} : S(x, t) = 0\}$ as a submanifold of the space-time $\mathbb{R}^n \times \mathbb{R}$. We shall assume that $\nabla S(x, t)|_{\Gamma_t} \neq 0$ for all fixed values of t , where $\nabla = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$. Let ν be the unit space normal to the surface Γ_t pointing in the positive direction such that $\frac{\partial S}{\partial x_j} = |\nabla S| \nu_j$, $j = 1, \dots, n$.

Let $f(x, t)$ be a function defined on the surface Γ_t for some time interval, and denote by $\frac{\delta f}{\delta t}$ the derivative with respect to time t as it would be computed by an observer moving with the surface. This derivative has the following geometrical interpretation. Let M_0 be a point on the surface at the time $t = t_0$. Construct the normal line to the surface at M_0 . At the time $t = t_0 + \Delta t$, Δt is sufficiently small, this normal meets the surface $\Gamma_{t_0 + \Delta t}$ at the point $M = M(t_0 + \Delta t)$. Then the δ -derivative is defined as

$$\frac{\delta f(M_0, t_0)}{\delta t} = \lim_{\Delta t \rightarrow 0} \frac{f(M) - f(M_0)}{\Delta t}. \quad (\text{A.1})$$

If Δs is the distance between M_0 and M , then

$$G = \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} \quad (\text{A.2})$$

is the *normal velocity of the moving surface* Γ_t and

$$\frac{\delta x_j}{\delta t} = \lim_{\Delta t \rightarrow 0} \frac{\Delta x_j}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\Delta S}{\Delta t} \frac{\Delta x_j}{\Delta S} = G \nu_j, \quad j = 1, \dots, n. \quad (\text{A.3})$$

Since it is essential that the δ -derivative is computed on a surface, and S remains constant on this surface, then $\frac{\delta S}{\delta t} = 0$. Thus we have

$$0 = \frac{\delta S}{\delta t} = \frac{\partial S}{\partial t} + \sum_{j=1}^n \frac{\delta S}{\delta x_j} \frac{\delta x_j}{\delta t} = \frac{\partial S}{\partial t} + \sum_{j=1}^n G |\nabla S| \nu_j^2,$$

i.e.,

$$S_t = -G |\nabla S|. \quad (\text{A.4})$$

From this formula we can see that $-G = \frac{S_t}{|\nabla S|}$ can be interpreted as the time component of the normal vector.

The space-time unit normal to the surface Γ is given by $\mathbf{n} = \frac{(\nu, -G)}{\sqrt{1+G^2}}$, where $\sqrt{1+G^2} = \frac{|\nabla_{(x,t)} S|}{|\nabla S|}$, $\nabla_{(x,t)} = (\nabla, \frac{\partial}{\partial t})$.

If $f(x, t)$ is a function defined only on Γ , its first order δ -derivatives with respect to the time and space variables are defined by the following formulas [12, 5.2.(15),(16)]:

$$\frac{\delta f}{\delta t} \stackrel{\text{def}}{=} \frac{\partial \tilde{f}}{\partial t} + G \frac{\partial \tilde{f}}{\partial \nu}, \quad \frac{\delta f}{\delta x_j} \stackrel{\text{def}}{=} \frac{\partial \tilde{f}}{\partial x_j} - \nu_j \frac{\partial \tilde{f}}{\partial \nu}, \quad j = 1, \dots, n, \quad (\text{A.5})$$

where \tilde{f} is a smooth extension of f to a neighborhood of Γ in $\mathbb{R}^n \times \mathbb{R}$, $j = 1, \dots, n$, and $\frac{\partial \tilde{f}}{\partial \nu} = \nu \cdot \nabla \tilde{f}$ is the normal derivative. Since δ -derivatives are independent of the way of extension of the function f , we shall drop tilde from the function f . Thus the gradient tangent to the surface Γ_t is defined as

$$\nabla_{\Gamma_t} = \nabla - \nabla_\nu = \left(\frac{\delta}{\delta x_1}, \dots, \frac{\delta}{\delta x_n} \right), \quad (\text{A.6})$$

where $\nabla_\nu = \nu(\nu \cdot \nabla)$ is the gradient along the normal direction to Γ_t . The *mean curvature* of the surface Γ_t is defined as

$$\mathcal{K} \stackrel{\text{def}}{=} -\frac{1}{2} \nabla_{\Gamma_t} \cdot \nu = -\frac{1}{2} \sum_{j=1}^n \frac{\delta \nu_j}{\delta x_j} = -\frac{1}{2} \nabla \cdot \nu. \quad (\text{A.7})$$

A.2. Distributions defined on a surface. The Heaviside function $H(S)$ is introduced by the following definition:

$$\langle H(S), \varphi(\cdot, \cdot) \rangle = \int_{S \geq 0} \varphi(x, t) dx dt, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R}).$$

According to [12, 5.3.(1),(2)], we introduce the delta function $\delta(S)$ on the surface Γ :

$$\langle \delta(S), \varphi(\cdot, \cdot) \rangle = \int_{-\infty}^{\infty} \int_{\Gamma_t} \varphi(x, t) d\Gamma_t dt = \int_{\Gamma} \varphi(x, t) \frac{d\Gamma}{\sqrt{1+G^2}}, \quad (\text{A.8})$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$, where $d\Gamma_t$ and $d\Gamma$ are the surface measures on the surfaces Γ_t and Γ , respectively. According to [12, 5.5.Theorem 1.], we have

$$\frac{\partial H(S)}{\partial x_j} = \nu_j \delta(S), \quad \frac{\partial H(S)}{\partial t} = -G \delta(S). \quad (\text{A.9})$$

A.3. An integration-by-parts formula. We need the following integrating-by-parts formula.

Lemma A.1. ([29, Lemma 9.1.], cf. [12, 5.2.(25),(26)]) *Suppose that $a(x, t)$ is a smooth function defined only on the surface $\Gamma = \{(x, t) : S(x, t) = 0\}$ which is the restriction of some smooth function defined in a neighborhood of Γ in $\mathbb{R}^n \times \mathbb{R}$, and $\Gamma_0 = \{x : S(x, 0) = 0\}$. Then the following formula for integration by parts holds:*

$$\int_{\Gamma} a \frac{\delta \varphi}{\delta t} \frac{d\Gamma}{\sqrt{1+G^2}} = - \int_{\Gamma} \frac{\delta^* a}{\delta t} \varphi \frac{d\Gamma}{\sqrt{1+G^2}} - \int_{\Gamma_0} a(x, 0) \varphi(x, 0) d\Gamma_0, \quad (\text{A.10})$$

for any $\varphi \in \mathcal{D}(\mathbb{R}^n \times [0, \infty))$, where $\frac{\delta^*}{\delta t}$ is the adjoint operator defined as

$$\frac{\delta^* a}{\delta t} = \frac{\delta a}{\delta t} - 2\mathcal{K}Ga = \frac{\delta a}{\delta t} + \nabla_{\Gamma_t} \cdot (aG\nu), \quad (\text{A.11})$$

\mathcal{K} is the mean curvature (A.7) of the surface Γ_t .

A.4. Transport theorems. Here we give the following transport theorems.

Theorem A.1. ([12, 12.8.(3)]) *Let $f(x, t)$ be a sufficiently smooth function defined in a moving solid Ω_t , and let a moving hypersurface $\partial\Omega_t$ be its boundary. Let ν be the outward unit space normal to the surface $\partial\Omega_t$ and $V(x, t)$ be the velocity of the point x in Ω_t . Then the volume transport theorem holds:*

$$\begin{aligned} \frac{d}{dt} \int_{\Omega_t} f(x, t) dx &= \int_{\Omega_t} \frac{\partial f}{\partial t} dx + \int_{\partial\Omega_t} fV \cdot \nu d\Gamma_t \\ &= \int_{\Omega_t} \left(\frac{\partial f}{\partial t} + \operatorname{div}(fV) \right) dx, \end{aligned} \quad (\text{A.12})$$

where $d\Gamma_t$ is the surface measure on the moving surface $\partial\Omega_t$.

Theorem A.2. ([12, 12.8.(9)]) *If $e(x, t)$ is a smooth function defined only on the moving surface $\Gamma_t = \{x : S(x, t) = 0\}$ (which is the restriction of some smooth function defined in a neighborhood of Γ_t), then the surface transport theorem holds:*

$$\begin{aligned} \frac{d}{dt} \int_{\Gamma_t} e(x, t) d\Gamma_t &= \int_{\Gamma_t} \left(\frac{\delta e}{\delta t} - 2\mathcal{K}Ge \right) d\Gamma_t = \int_{\Gamma_t} \left(\frac{\delta e}{\delta t} + \nabla_{\Gamma_t} \cdot (eU_{\delta}) \right) d\Gamma_t, \end{aligned} \quad (\text{A.13})$$

where $U_{\delta} = \nu G$ is the velocity of Γ_t given by (2.3).

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